Chain Rule, Directional Derivatives, and Optimization

1. Chain Rule

Consider the function $z = f(x,\ y)$ where (x, y) is a function of u, v, and w, i.e., $(x,\ y) = g(u,\ v,\ w)$. (Note that $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$, so $f \circ g : \mathbb{R}^3 \to \mathbb{R}$.) Suppose that we want to know the partial derivatives of z with respect to u, v, and w.

(a) Draw a tree diagram to understand how z depends on each variable. Use this to find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$, and $\frac{\partial z}{\partial w}$.

(c) If, for example, $z = x^2 y$, $x = u + v$, and $y = uvw^3$, find ∇z.

(d) Check your answer to (c) by substituting expressions for x and y into $z = x^2 y$ and then differentiating.

2. Directional Derivative

Suppose that over a certain region of space, the electrical potential V is given by $V(x,\ y, z) = 5x^2 + 3xy + xyz$.

(a) Find the rate of change of the potential at $P = (3, 4, 5)$ in the direction of vector $v = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

(b) In what direction does V change most rapidly at P?

(b) What is the maximum rate of change at P?

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3. Critical Points and the Second Derivative Test

A simple case of constrained optimization is finding the local max/min of the function $f(\vec{x})$ subject to the constraint $g(\vec{x}) = c$. Use your knowledge of critical points and the second derivative test to minimize $x^2 + y^2 + z^2$ subject to $x + 2y + z = 6$. (Recall: critical points $\vec{a}$ satisfy $\vec{\nabla} f(\vec{a}) = \vec{0}$.)

4. Lagrange multipliers

Now let's consider the same problem with Lagrange multipliers. One interpretation is that the Lagrange multiplier represents a penalty. Then the problem minimize $f(\vec{x})$ such that $g(\vec{x}) = c$ can be thought of as minimizing $f(\vec{x}) + \lambda(c - g(\vec{x}))$. By setting the gradient of this system equal to zero, we find that critical points satisfy

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g, \\ g(\vec{x}) = \vec{c}. \end{cases}$$

Use this to minimize $x^2 + y^2 + z^2$ subject to $x + 2y + z = 6$.